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INVERSION OF THE RADON TRANSFORM, BASED ON THE THEORY OF A-ANALYTIC FUNCTIONS, WITH APPLICATION TO 3D INVERSE KINEMATIC PROBLEM WITH LOCAL DATA

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Abstract - In the introduction we show that the inverse problems for transport equations are naturally reduced to the Cauchy problem for the so called A-analytic functions, and hence the solution is given in terms of operator analog of the Cauchy transform. In section 1 we develop elements of the theory of A-analytic functions and obtain stability estimates for our Cauchy transform. In section 2 we discuss numerical aspects of this transformation. In section 3 we apply this algorithm to the 3-dimensional inverse kinematic problem with local data on the Earth surface, using modified Newton method and discuss numerical examples.

1. INTRODUCTION

Let's consider a 2D stationary transport equation:

$$Pu = \langle \omega, \nabla_x u(x,\alpha)\rangle + \mu(x)u(x,\alpha) = a(x),\ x \in \Omega,\ \alpha \in \mathbb{R},$$

where $\omega = (\cos\alpha, \sin\alpha)$, $u(x,\alpha)$ — a 2π-periodic function, describing the density of particles at a point x, moving in a direction ω, $\mu(x)$ — a function, defining the attenuation at a point x, $a(x)$ — the radiation source. This problem is considered in a strictly convex domain Ω with a smooth boundary. If $x \in \partial\Omega$, then one can measure the incoming and outgoing flow of particles.

$$u(x,\omega)\Big|_{\Sigma} = f(x,\omega) = \begin{cases} f_-(x,\omega), & \langle\omega,\nu\rangle < 0 \\ f_+(x,\omega), & \langle\omega,\nu\rangle \geq 0, \end{cases}$$

where f is a 2π-periodic function, f_- — the incoming (into the domain Ω) particle flow, f_+ — the outgoing (from Ω) particle flow, ν — the outer unit normal to the Ω, and $\Sigma = \partial\Omega \times [0, 2\pi]$, which can be identified with the torus $\partial\Omega \times S^1$, $S^1 = \{\omega \in \mathbb{R}^2;\ |\omega| = 1\}$ in a natural way.

The inverse problem consists in determining the right-hand side a from the given trace of u on Σ and attenuation function μ. For the solution of this inverse problem it is useful to rewrite it in complex variables, assuming that: $z = x_1 + ix_2$, $i^2 = -1$, $u(x,\alpha) = u(z,\alpha)$, $f(x,\alpha) = f(z,\alpha)$, $\mu(x) = \mu(z)$. Then using Euler's formulae one gets

$$e^{i\alpha}Pu = \overline{\partial}u + e^{2i\alpha}\partial u + e^{i\alpha}\mu u = e^{i\alpha}a,$$

where the formal derivatives $\partial = \frac{\partial}{\partial z}$ and $\overline{\partial} = \frac{\partial}{\partial \overline{z}}$ are defined in the usual way:

$$\partial = \frac{\partial}{\partial z} = \frac{\partial_1 - i\partial_2}{2},\ \overline{\partial} = \frac{\partial}{\partial \overline{z}} = \frac{\partial_1 + i\partial_2}{2}.$$

The expansion of the function u into the Fourier series has the form:

$$u(x,\alpha) = \sum_{n\in\mathbb{Z}} u_n e^{-in\alpha}.$$

Here $u_n = \widehat{u}_{-n}$, where $\widehat{u}_n$ are the usual Fourier coefficients.

Since the function u is real-valued, one gets $u_n = \bar{u}_{-n}$ and hence, setting $\mathbf{u} = (u_0, u_1, u_2, \ldots) \in l_2(0,\infty)$ and introducing the shift operator U,

$$U : (u_0, u_1, u_2, \ldots) \mapsto (0, u_0, u_1, u_2, \ldots),$$

and its adjoint in $l_2(0,\infty)$, one can rewrite this inverse problem as the Cauchy problem

$$\begin{cases} \overline{\partial}_A \mathbf{u} + \mu U^* \mathbf{u} = 0 \quad \text{in } \Omega, \\ \mathbf{u}\Big|_{\partial\Omega} = \mathbf{f}, \end{cases} \tag{1}$$

where $A = -(U^*)^2$, and $\overline{\partial}_A = \overline{\partial} - A\partial$ is a Beltrami-type operator with the operator coefficient A. However, in the case under consideration in any of the Hilbert spaces $l_2^s(0,\infty)$ with the norm

$$||\mathbf{u}||_s^2 = \sum_{n=0}^{\infty}(1+n^2)^s |u_n|^2, \quad s \geq 0,$$

the norm of the operator A exactly equals to 1.

This makes the theory of equation $\overline{\partial}_A \mathbf{u} = 0$ more difficult than in the classic case when the norm of A is less than 1. A solution of the equation $\overline{\partial}_A \mathbf{u} = 0$ is called an A-analytic function.

We can cancel the term $\mu(z)U^*\mathbf{u}$ by change of variables. Putting

$$\mathbf{u}(z) = e^{\Phi(z)}\mathbf{v}(z),$$

we find that the operators $\overline{\partial}_A + \mu U^*$ and $\overline{\partial}_A$ are similar, provided that the family of operators Φ satisfy the equation

$$\overline{\partial}_A \Phi = -\mu(z) U^*, \quad z \in \Omega. \tag{2}$$

Under this assumption the problem (1) reduces to the problem

$$\begin{cases} \overline{\partial}_A \mathbf{v} = \mathbf{0} \quad \text{in } \Omega, \\ \mathbf{v}\Big|_{\partial\Omega} = e^{-\Phi}\mathbf{u}\Big|_{\partial\Omega} = e^{-\Phi}\mathbf{f} := \mathbf{g}. \end{cases} \tag{3}$$

The existence and boundedness of this transformation operator is proved in the paper [1].

The solution to the Cauchy problem (3) can be obtained by the analog of the Cauchy formula for the operator $\overline{\partial}_A$, also derived in [1]:

$$\mathbf{v}(z) = \frac{1}{2\pi i}\int_{\partial\Omega} \mathcal{C}(\zeta - z)(d\zeta + A d\overline{\zeta})\mathbf{v}(\zeta) - \frac{1}{\pi}\int_{\Omega} \mathcal{C}(\zeta - z)\overline{\partial}_A \mathbf{v}(\zeta) d\zeta, \tag{4}$$

where $\zeta = \zeta_1 + i\zeta_2$, $z = x_1 + ix_2$ and

$$\mathcal{C}(z) = (z + \overline{z}A)^{-1} = z^{-1} \cdot \left(I + \frac{\overline{z}}{z}A\right)^{-1}$$

is the operator analog of the Cauchy kernel. With the help of this formula the solution of the problem (3) is written in the explicit form:

$$\mathbf{v}(z) = \frac{1}{2\pi i}\int_{\partial\Omega} \mathcal{C}(\zeta - z)(d\zeta + A d\overline{\zeta})\mathbf{g}(\zeta).$$

It's known that $\mathcal{C}(z) \in \mathcal{L}(l_2^{s+1}, l_2^s)$ for all $s > -1/2$, and the operator $\Phi(z) \in \mathcal{L}(l_2)$ can also be found from the Cauchy formula (4)

$$\Phi(z) = \frac{1}{\pi}\int_{\Omega} \mathcal{C}(\zeta - z)\mu(\zeta)U^* d\zeta, \ \mu \in C^2(\overline{\Omega}).$$

Combining these formulae, one gets the final formula for restoration of the function a :

$$a(z) = 2\mathrm{Re}\{(u_1)_z\} + \mu(z)u_0(z), \tag{5}$$

where

$$u_m(z) = \langle \mathbf{u}(z), \mathbf{e}_m \rangle, \quad m = 0, 1, \ldots, \tag{6}$$

$\mathbf{e}_m = (\underbrace{0,\ldots,0}_{m},1,0,\ldots,0)$, $\langle\cdot,\cdot\rangle$ is a scalar product in $l_2(0,\infty)$, and

$$\mathbf{u}(z) = \frac{1}{2\pi i}\int_{\partial\Omega} \mathcal{C}(\zeta - z)e^{\Phi(z)-\Phi(\zeta)}(d\zeta + A d\overline{\zeta})\mathbf{f}(\zeta). \tag{7}$$

Notice that the operator $\Phi(z)$ is defined non-uniquely, in particular $\Phi(z) + \Phi_0(z)$, where $\Phi_0 \in C^1(\overline{\Omega}; \mathcal{L}(l_2))$ and $\overline{\partial}_A \Phi_0 = 0$, also satisfies the equation (2).

Formulae (5)–(7) give the solution of the ill-posed inverse problem in terms of the Fourier coefficients of the initial function $f(x, \alpha)$. Using the Parseval equality, these formulae can be rewritten in terms of the function f itself.

We see that from abstract point of view our inverse problem (with or without attenuation) is reduced to the problem of finding the vector $(I - A^*A)\mathcal{C}f$, where $\mathcal{C}$ is the A-analog of the Cauchy transform. We shall study the properties of this transform in the next section in more general abstract setting.

We can see that abstract operator formulae lead to *unsaturated* (see [2], [3] and the remark at the end of the section 3) and easily implemented algorithms.

2. A-ANALYTIC FUNCTIONS

Let X be a complex Hilbert space and A be a linear bounded operator acting in X. For natural $n = 1, 2, \ldots$ we put $X_n = \{u \in X,\ A^n u = 0\}$ and $X_0 = \bigcup_n X_n$. We suppose that linear subspace X_0 is dense in X, and for any $u \in X$

$$||Au|| \le ||u||, \tag{8}$$

$$||A^n u|| \to 0 \quad \text{when } n \to \infty. \tag{9}$$

For any $s \in \mathbb{R}$ we put

$$||u||_s^2 = \sum_{n=0}^{\infty} (1 + n^2)^s ||(I - A^*A)^{1/2} A^n u||^2. \tag{10}$$

X^s is a closure in this norm of the linear space X_0. Similarly, $X^{s,1/2}$ is a closure of X_0 in the norm

$$||u||_{s,1/2}^2 = \sum_{n=0}^{\infty} (1 + n^2)^s ||A^n u||^2 \tag{11}$$

It is easy to check that

$$X^0 = X \subset X^{-s,1/2} \quad \text{if } s > 1/2$$

and

$$X^{s+1/2} \cong X^{s,1/2} \quad \text{if } s > -1/2.$$

These properties explain our notations.

Remark 1.1 If we change operator A to operator αA where α is a unit complex number, then norms (10)–(11) will remain the same. Since under a conformal map f the operator $\overline{\partial}_A$ goes (up to a nonzero factor $\overline{f}'(z)$) to $\overline{\partial}_{\alpha A}$ with $|\alpha| = 1$, we can say that these scales are conformally invariant. The following example is typical for our applications.

Example 1.2 Let $X = l_2((0, \infty); Y)$ be the space of sequences $u = (u_0, u_1, u_2, \ldots)$, whose values lie in an auxiliary Hilbert space Y,

$$||u|| = \sum_{n=0}^{\infty} |u_n|^2,$$

where $|\cdot|$ is a norm in Y. Operator A is a left shift

$$Au = (u_1, u_2, \ldots).$$

Then $X^s = l_2^s((0, \infty); Y)$ with norm

$$||u|| = \sum_{n=0}^{\infty} (1 + n^2)^s |u_n|^2.$$

For this example it is easy to check that if we replace the operator A with the operator A^n the norms in our new scales will be equivalent to the norms given by (10) and (11). Operator $(I - A^*A) = (I - A^*A)^{1/2}$ is the orthogonal projector on the space $(Y, 0, 0, \ldots)$ which we identify with Y. Similarly $(I - (A^*)^n A^n)$ is the orthogonal projector on space $(\underbrace{Y, Y, \ldots, Y}_{n \text{ times}}, 0, 0, \ldots)$ which we identify with Y^n. Let $R(\lambda, A) = (\lambda - A)^{-1}$

be the resolvent of our operator A. Conditions (8) and (9) yield that $R(\lambda, A)$ is a bounded operator in X for $|\lambda| > 1$ and is in general unbounded for $|\lambda| = 1$. Next theorem shows that $R(\lambda, A) \in \mathcal{L}(X^{s+1}, X^s)$ on the unit circle $S = \{\lambda \in \mathbb{C}, |\lambda| = 1\}$.

Theorem 1.3 For all $\lambda \in S$ and for $s > -1/2$ the following estimate holds

$$||R(\lambda, A)u||_s \leq \frac{c(s)}{(s + 1/2)} ||u||_{s+1},$$

where $c(s)$ is a finite constant for all $s \geq -1/2$.

Let us now study the properties $R(\lambda, A)$ as an operator-valued function on the unit circle S, puting $\lambda = e^{i\varphi}$. If $f(\varphi)$ is a 2π-periodic smooth function with vector or operator values, the Weyl's derivatives $\mathcal{D}^\alpha_\pm$ are given by formula

$$\mathcal{D}^\alpha_\pm f = \sum (\pm in)^\alpha \widehat{f}_n e^{in\varphi}, \quad \alpha \geq 0. \tag{12}$$

Here $\widehat{f}_n$ is the Fourier coefficient of function f and $(in)^\alpha = |n|^\alpha e^{\mathrm{sign}(n)\alpha\pi i/2}$. Formula (12) will be true also for $\alpha < 0$, if

$$\widehat{f}_0 = 0, \tag{13}$$

and we have $\mathcal{D}^{-\alpha}_\pm \mathcal{D}^\alpha_\pm = I$ — is the identity operator for $\alpha \geq 0$ and also for $\alpha < 0$ on the subspace (13). We also have the following integration by parts formula:

$$\int_0^{2\pi} \mathcal{D}^\alpha_+ u(\varphi) v(\varphi) d\varphi = \int_0^{2\pi} u(\varphi) \mathcal{D}^\alpha_- v(\varphi) d\varphi$$

Here u is an operator-valued function. Using these formulae it is easy to prove

Theorem 1.4 For all $u \in X$ the following estimates hold

$$||\mathcal{D}^{-\alpha}_+ Ru|| \leq \zeta(\alpha)||u||, \quad \alpha > 1,$$

$$||\mathcal{D}^{-\alpha}_+ Ru||^2_{L_2(S;X)} \leq \zeta(2\alpha)||u||^2, \quad \alpha > 1/2.$$

Here $\zeta(\alpha)$ is the Riemann's zeta function. As a corollary we obtain

$$\mathcal{D}^{-\alpha-s}_+(Ru) \in H^s(S; X), \quad \alpha > 1/2$$

and by Sobolev embedding theorem $\mathcal{D}^{-\alpha-n-1}_+(Ru) \in C^n(S; X)$, $\alpha > 1/2$. Here H^s is the Sobolev space with values in X.

Let Ω be an open set in the complex plane $\mathbb{C}$. As in the introduction we say that function $u(z)$ with vector or operator values is A-analytic in Ω iff $\overline{\partial}_A u \equiv (\overline{\partial} - A\partial)u = 0$ for all $z \in \Omega$. Now we would like to describe the main a priori estimates and identities from which it is easy to obtain correctness for the corresponding boundary value problems for operator $\overline{\partial}_A$. To do this we suppose for simplicity that Ω is a simply connected bounded domain with smooth boundary $\partial\Omega = \{z = z(s), s \in [0, l]\}$, where s is the natural parameter and l is the length of $\partial\Omega$. On the space of l-periodic functions on $\partial\Omega$ we introduce the Hilbert transform $\mathcal{H}$ (which just multiplies each n-th Fourier coefficient by $-i \cdot \mathrm{sign}(n)$) and a self-adjoint non-negative operator

$$\Lambda = \frac{l}{2\pi} \mathcal{H} \frac{d}{ds}.$$

We put

$$u_+ = \frac{(I + i\mathcal{H})u}{2}, \quad u_- = \frac{(I - i\mathcal{H})u}{2}.$$

Theorem 1.5
(i) If u is A-analytic in Ω and $u_+ \in H^{1/2}(\partial\Omega; X)$ then the following identity holds:

$$2||(I - A^*A)^{1/2}\partial u||^2_{L_2(\Omega;X)} + ||\Lambda^{1/2} u_-||^2_{L_2(\partial\Omega;X)} = ||\Lambda^{1/2} u_+||^2_{L_2(\partial\Omega;X)}. \tag{14}$$

(ii) If u is A-analytic and for some real $s \geq -1/2$, $u_+ \in H^{1/2}(\partial\Omega; X^{s,1/2})$ then the following identity holds:

$$2||\partial u||^2_{L_2(\Omega;X^s)} + ||\Lambda^{1/2}u_-||^2_{L_2(\partial\Omega;X^{s,1/2})} = ||\Lambda^{1/2}u_+||^2_{L_2(\partial\Omega;X^{s,1/2})}.$$

If we introduce an A-analog of the Cauchy transform

$$(\mathcal{C}f)(z) = \frac{1}{2\pi}\int\limits_{\partial\Omega} \nu_A(\zeta - z)^{-1}_{-A} f(\zeta)ds,$$

where ν is a unit outward normal (to Ω) in complex form $\nu = \nu_1 + i\nu_2$ and $\nu_A = \nu - A\overline{\nu}$, we obtain from Theorem 1.5 the following

Theorem 1.6 The following estimates hold:

$$||(I - A^*A)^{1/2}\mathcal{C}f||_{H^1_{(\Omega;X)}} \leq c||f||_{H^{1/2}(\partial\Omega;X)}, \tag{15}$$

$$||\mathcal{C}f||_{H^1(\Omega;X^s)} \leq c(s)||f||_{H^{1/2}(\partial\Omega;X^{s,1/2})} \quad \text{for all} \quad s > -1/2.$$

Formulae (14) and (15) show the stability of the inversion of the Radon transform based on the A-analog of the Cauchy transformation.

3. CAUCHY FORMULA IN POLAR COORDINATES; ESTIMATION OF THE CONVERGENCE SPEED OF THE PROJECTION METHOD

In this section we will rewrite the Cauchy formula (7) in polar coordinates and will estimate the error that occurs after the Fourier coefficients with indices $\geq 2N+1$ are discarded. For simplicity we'll consider the case $\mu = 0$. Then

$$a(z) = 2\text{Re}(u_1)_z, \tag{16}$$

$$\mathbf{u}(z) = \frac{1}{2\pi i}\int_{\partial\Omega} \mathcal{C}(\zeta - z)(d\zeta + Ad\overline{\zeta})\mathbf{f}(\zeta). \tag{17}$$

So, let Ω be a convex domain with a smooth boundary, $z \in \Omega$, $\zeta \in \partial\Omega$. Let's switch to the polar coordinate system with the centre at the point z (see Figure 1),

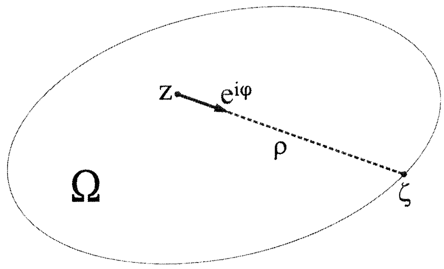

Figure 1: Polar coordinate system with the centre at the point z.

$$\zeta = z + \rho(\varphi, z)e^{i\varphi}, \ \rho > 0.$$

Then, using identities

$$\left(I + e^{-2i\varphi}A\right)^{-1}(\zeta - z)^{-1}\left(d\zeta + Ad\overline{\zeta}\right) = \begin{cases} \left(I + e^{-2i\varphi}A\right)^{-1} 2id\varphi + \frac{d\zeta}{\zeta - z} \\ -\left(I + e^{-2i\varphi}A\right)^{-1} e^{-2i\varphi}A2id\varphi + \frac{d\overline{\zeta}}{\overline{\zeta - z}} \end{cases}$$

the Cauchy formula (17) can be written in two forms:

$$\mathbf{u}(z) = \frac{1}{\pi}\int_0^{2\pi}\left(I + e^{-2i\varphi}A\right)^{-1}\mathbf{f}d\varphi + \frac{1}{2\pi i}\int_{\partial\Omega}\frac{\mathbf{f}(\zeta)d\overline{\zeta}}{\overline{\zeta - z}}, \tag{18}$$

$$\mathbf{u}(z) = -\frac{1}{\pi}\int_0^{2\pi} \left(I + e^{-2i\varphi}A\right)^{-1} e^{-2i\varphi}A\mathbf{f}d\varphi + \frac{1}{2\pi i}\int_{\partial\Omega}\frac{\mathbf{f}(\zeta)d\zeta}{\zeta - z}. \tag{19}$$

We have

$$\frac{d\zeta}{\zeta - z} + \frac{d\overline{\zeta}}{\overline{\zeta - z}} = 2\frac{\rho'}{\rho}d\varphi,$$

whence, adding together formulae (18), (19) and dividing the result by 2, we get

$$\mathbf{u}(z) = \frac{1}{2\pi}\int_0^{2\pi} \left(I + e^{-2i\varphi}A\right)^{-1}\left(I - e^{-2i\varphi}A\right)\mathbf{f}(\zeta)d\varphi + \frac{1}{2\pi i}\int_0^{2\pi}\mathbf{f}\cdot\frac{\rho'_\varphi}{\rho}d\varphi.$$

In particular, if Ω is a circle with the centre at the point z, then $\rho'_\varphi = 0$ and we get a mean-value theorem

$$\mathbf{u}(z) = \frac{1}{2\pi}\int_0^{2\pi} \left(I + e^{-2i\varphi}A\right)^{-1}\left(I - e^{-2i\varphi}A\right)\mathbf{f}d\varphi.$$

For finding the function a by the formula (16), it's convenient to use the formula (18), as in this case the second term vanishes under the action of the operator ∂:

$$\partial\mathbf{u}(z) = \partial\frac{1}{\pi}\int_0^{2\pi} \left(I + e^{-2i\varphi}A\right)^{-1}\mathbf{f}(\zeta(z,\varphi))d\varphi. \tag{20}$$

Recalling, that $A = -(U^*)^2$, we get

$$\left(I + e^{-2i\varphi}A\right)^{-1} = \sum_{n=0}^{\infty} e^{-2ni\varphi}(U^*)^{2n}.$$

Since $\left\langle (U^*)^{2n}\mathbf{f}, \mathbf{e}_1\right\rangle = \left\langle \mathbf{f}, U^{2n}\mathbf{e}_1\right\rangle = \langle \mathbf{f}, \mathbf{e}_{2n+1}\rangle = f_{2n+1}$, where $\mathbf{e}_1 = (0,1,0,\ldots,0)$, $\langle\cdot,\cdot\rangle$ is a scalar product in $l_2(0,\infty)$, from (16), (20) we get

$$a(z) = 2\mathrm{Re}\partial\frac{1}{\pi}\int_0^{2\pi}\sum_{n=0}^{\infty} e^{-2ni\varphi} f_{2n+1}\Big(\underbrace{z + \rho(z,\varphi)e^{i\varphi}}_{\zeta(z,\varphi)}\Big)d\varphi$$

or, performing the differentiation under the integral sign,

$$a(z) = 2\mathrm{Re}\frac{1}{\pi}\int_0^{2\pi}\sum_{n=0}^{\infty} e^{-2ni\varphi} f'_{2n+1}(\zeta)\zeta'_z d\varphi.$$

Put

$$a_N(z) = 2\mathrm{Re}\frac{1}{\pi}\int_0^{2\pi}\sum_{n=0}^{N-1} e^{-2ni\varphi} f'_{2n+1}(\zeta)\zeta'_z d\varphi \tag{21}$$

and let $\overline{\Omega}' \subset \Omega$ be a compact subdomain in Ω. Then, if $\partial\Omega \in C^1$, then $|\zeta'_z| \le \mathrm{const}\ \forall z \in \overline{\Omega}'$ and

$$||a - a_N||_{C(\overline{\Omega}')} = o(N^{-s+\frac{1}{2}}) \quad \forall s > \frac{1}{2} \tag{22}$$

when $N \to \infty$, if $\mathbf{f}'_\zeta \in l_2^s(0,\infty)$ uniformly in $\zeta \in \partial\Omega$. So, smoother initial data $\mathbf{f}$ you have, less Fourier coefficients are required for reconstructing the function a with the prescribed accuracy. The estimate (22) also shows, that the projection method (21) bears *unsaturated* character (see [2], [3]).

4. LOCAL 3D INVERSE KINEMATIC PROBLEM ON REFRACTED RAYS

Let's consider a half-space in 3D and let

$$x^0 = (x_1^0, x_2^0) = r(\cos\alpha, \sin\alpha),$$

$$x^1 = (x_1^1, x_2^1) = r(\cos\beta, \sin\beta),$$

where $\alpha, \beta \in [0, 2\pi)$, $r \in (0, \rho]$, are two arbitrary points on the circle $|x| = r$ in the (x_1, x_2)-plane.

Let $\Gamma = \Gamma(n;\alpha,\beta,r) = \Gamma(x^0,x^1)$ be a ray, corresponding to the slowness $n(x,z)$, that passes through the points x^0 and x^1. We assume that for any pair of two points, x^0 and x^1, satisfying $|x^0| = |x^1| = r$, $r \in (0,\rho]$, there exists only one ray $\Gamma(x^0,x^1)$ connecting them. Suppose that we know the travel time

$$\tau(\alpha,\beta,r) = \int\limits_{\Gamma(n;\alpha,\beta,r)} nds := F(n), \tag{23}$$

and would like to find the slowness n. At the first step we reduce this nonlinear integral equation to a sequence of linear equations using the modified Newton's method. To this end, note that

$$\begin{aligned} F(n+h) - F(n) &= \int\limits_{\Gamma(n+h)} (n+h)ds - \int\limits_{\Gamma(n)} nds \\ &= \int\limits_{\Gamma(n+h)} [(n+h)-n]ds + \underbrace{\int\limits_{\Gamma(n+h)} nds - \int\limits_{\Gamma(n)} nds}_{o(h) \text{ by Fermat's principle}} \approx \int\limits_{\Gamma(n)} hds = F'(n)h, \end{aligned}$$

where we use the abbreviation $\Gamma(n;\alpha,\beta,r) \equiv \Gamma(n)$. Recall that the *Fermat principle* states that the travel time along the ray is stationary with respect to the small perturbations in the ray trajectory. After we rewrite the equation (23), $F(n) = \tau$, in the form

$$F(n) \equiv \underbrace{F(n_0+u) - F(n_0)}_{\approx F'(n_0)u} + \underbrace{F(n_0)}_{\tau_0} = \tau,$$

we get

$$F'(n_0)u = \tau - F(n_0) := g(\alpha,\beta,r) \tag{24}$$

and

$$u = \left[F'(n_0)\right]^{-1}\left(\tau - F(n_0)\right).$$

Note that this is a standard linearization. Now we use the modified Newton iteration method

$$\begin{aligned} \tau = F(n) &\equiv F(n_k + (n-n_k)) - F(n_k) + F(n_k) \\ &\approx \underbrace{F'(n_k)}_{\approx F'(n_0)}(n-n_k) + F(n_k) \approx F'(n_0)(n-n_k) + F(n_k), \end{aligned}$$

and hence for the next iteration $n_{k+1} := n$ we have

$$n_{k+1} = n_k - \left[F'(n_0)\right]^{-1}\left(F(n_k) - \tau\right).$$

On the next step we choose $n_0 = n_0(z)$ so that the calculation of $\left[F'(n_0)\right]^{-1}$ would lead to the sequence of usual 2D Radon transform inversions in the discs $|x| < r$, $r \in (0,\rho]$. In the case $n_0 = n_0(z)$ it is easy to see that $\Gamma(n_0)$ is determined by the system of equations

$$\begin{aligned} p_1 := \langle x,\nu\rangle - h &= 0, \\ p_2 := z - \varphi(|x|,r) &= 0, \end{aligned}$$

or $p(x,z) = 0$ in short, where we let $p = (p_1,p_2)$. For the linear velocity $V_0 = m + bz$ we have $n_0 = (m+bz)^{-1}$ and

$$\varphi = \sqrt{r^2 + \left(\frac{m}{b}\right)^2 - |x|^2} - \frac{m}{b}.$$

Also

$$F'(n_0)u = \int\limits_{\Gamma(n_0;\alpha,\beta,r)} uds = \int |\nabla p|\delta(p)u(x,z)dxdz, \tag{25}$$

where $\delta(p) = \delta(p_1)\delta(p_2)$ is a Dirac delta-function concentrated on the curve $p = 0$, where we need to calculate ∇p_j. Here we used the usual formula

$$\int \delta(p(x))u(x)dx = \int\limits_{p=0} \frac{u d\sigma}{|\nabla p|},$$

where $d\sigma$ is a surface element. In general, $p(x) = \big(p_1(x), p_2(x), \dots, p_k(x)\big)$, $k \leq n$, $x \in \mathbb{R}^n$, $u \in C_0^\infty(\mathbb{R}^n)$ and

$$|\nabla p| := \sqrt{\det\langle \nabla p_i, \nabla p_j \rangle},$$

$\langle \cdot, \cdot \rangle$ is the scalar product in $\mathbb{R}^n$ and ∇p_j should be linearly independent. In the case of linear sound velocity we have

$$w^2(x,h) := |\nabla p|^2 = 1 + {\varphi'_{|x|}}^2 \left(1 - \frac{h^2}{|x|^2}\right),$$

and so from (25) we conclude that

$$F'(n_0)u = \int\limits_{\Gamma(n_0;\alpha,\beta,r)} u d\sigma = \int |\nabla p| \delta(p) u(x,z) dx dz = \int |\nabla p| \delta(p_1)\delta(p_2) u(x,z) dx dz$$

$$= \int w(x,h)\delta(\langle x, \nu \rangle - h) u(x, \varphi(|x|, r)) dx = g(\alpha, \beta, r), \qquad (26)$$

with $g(\alpha, \beta, r)$ as defined in (24).

In order to obtain an inversion formula for (26), $F'(n_0)u = g$, expressed through the inversion formula of the Radon transform, it is necessary that the weight $w(x,h)$ allows the factorization

$$w(x,h) = w_1(|x|) \times w_2(h^2),$$

which takes place if and only if

$$n_0(z) = (m + bz)^{-1}$$

(see [6]).

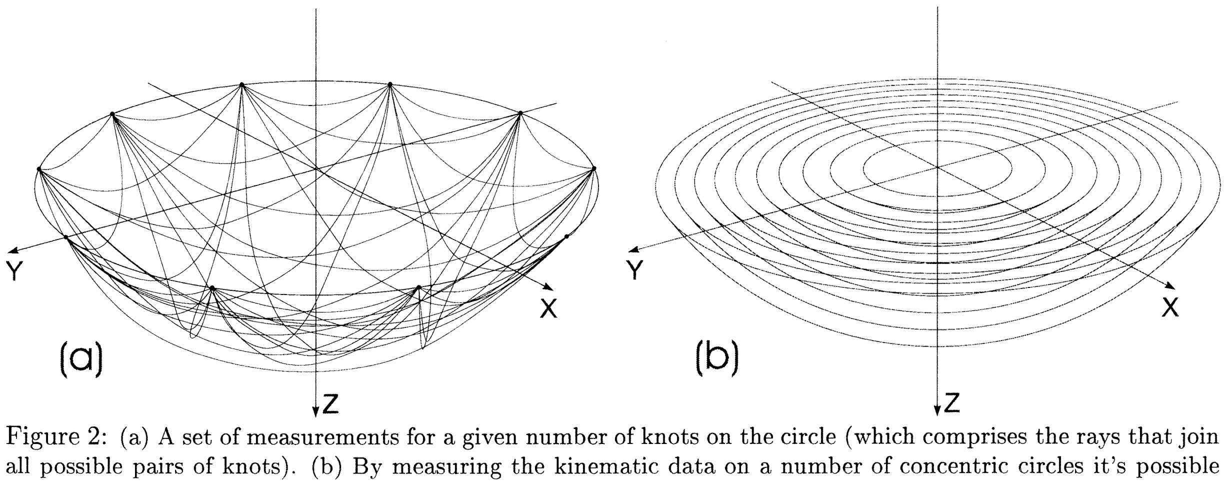

Figure 2: (a) A set of measurements for a given number of knots on the circle (which comprises the rays that join all possible pairs of knots). (b) By measuring the kinematic data on a number of concentric circles it's possible to reconstruct the slowness in the 3D volume.

We are primarily interested in the case of a *linear sound velocity*, where $n_0(z) = (m + bz)^{-1}$, $b > 0$, and

$$w^2(|x|, h) = \frac{r^2 + \left(\frac{m}{b}\right)^2 - h^2}{r^2 + \left(\frac{m}{b}\right)^2 - |x|^2}.$$

So, using (26), the solution $u(x,z)$ is given by the *inversion formula*

$$u(x,z) = \sqrt{r^2 + \left(\frac{m}{b}\right)^2 - |x|^2} \times \mathcal{R}^{-1}\left(\frac{g}{\sqrt{r^2 + \left(\frac{m}{b}\right)^2 - h^2}}\right), \qquad (27)$$

where $\mathcal{R}^{-1}$ is the 2D *inverse Radon transform.* We determine the unknown function u from the inversion formula (27) in a disc $x_1^2 + x_2^2 \leq r^2$, thus getting the sought-for refractive index (or slowness) $n_1(x, z) = n_0(z) + u(x, z)$ on the surface of a spherical segment, supported by a circle $x_1^2 + x_2^2 = r^2$, see Figure 2.

Numerical results show that the second iteration of a nonlinear algorithm can be better than the linearized approximation (see Figure 3). On each iteration we determine the slowness on the set of points on several spherical layers, and then smoothly interpolate it using radial basic functions in the half-space. It is required for ray tracing when solving the direct problem on next iteration. Algorithm also shows good results in the case of non-regular metrics (as on the Figure 4).

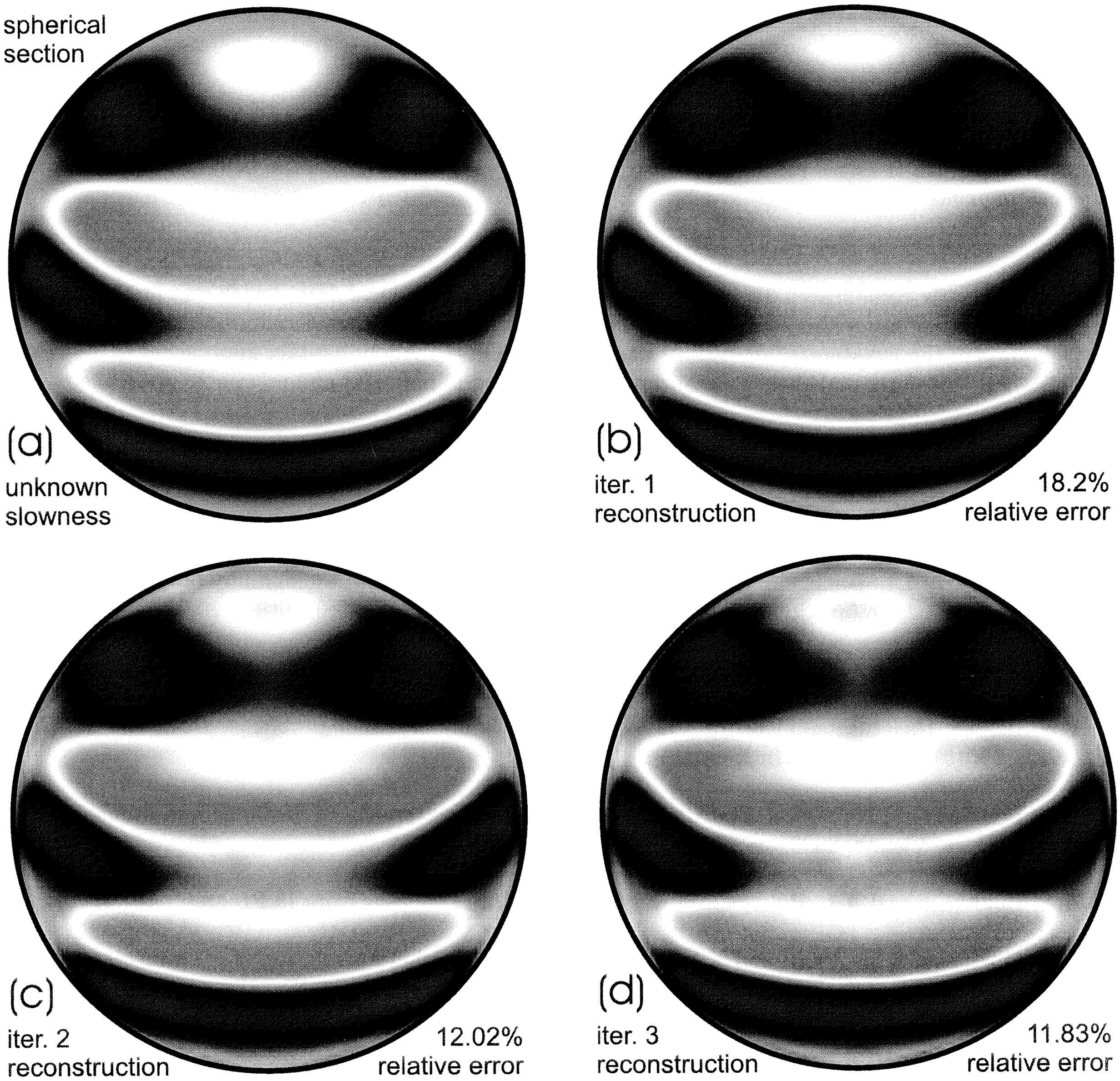

Figure 3: Examples of reconstruction after several iterations. (a) Original *small additive* of the slowness u, which is the difference between the original unknown slowness n and the *background* known slowness $n_0 = (m + bz)^{-1}$ is shown in a spherical section. (b) Its reconstruction from 128 projections after 1 iteration. (c) Reconstruction after 2 iterations. (d) Reconstruction after 3 iterations. The L_2-norm of a *small additive* u constitutes only 2% of the *background* slowness n_0, so we got a good reconstruction already after 1 iteration. But it refines on successive iterations. Relative errors were computed in the L_2-norm.

5. CONCLUSIONS AND REMARKS

In the introduction we reminded a known complex interpretation of inverse problems for transport equation (see [1]and references given there). The purpose of sections 2–3 was to show that the inversion of the Radon (or generalized Radon transform) in terms of the A-analog of the Cauchy transform is not only elegant but also leads to very simple and stable algorithms. In order to demonstrate this in section 4 we

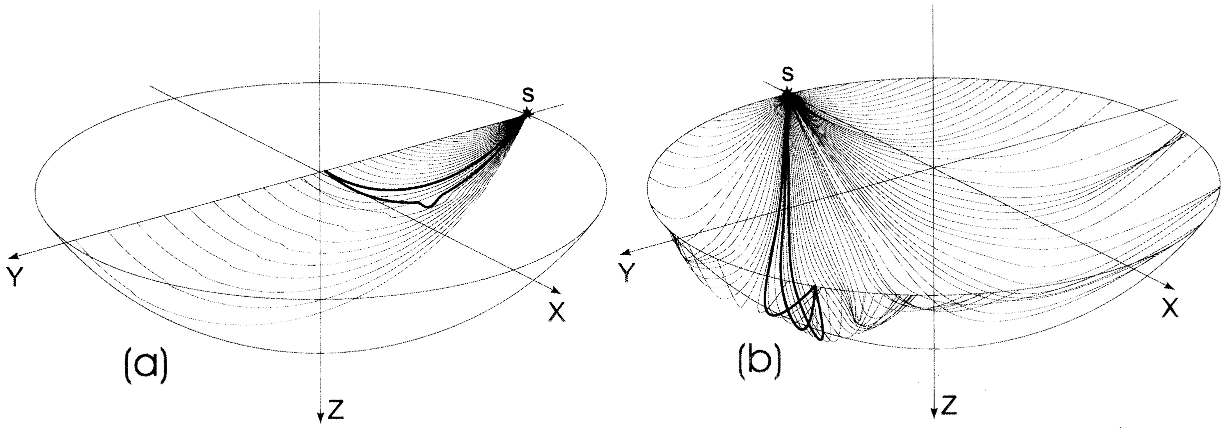

Figure 4: Examples of nonregular media. **s** denotes a source point. (a) **Two** different geodesics (shown in bold-faced type) join the same pair of points. (b) **Three** different geodesics (shown in bold-faced type) join the same pair of points.

chose a sufficiently complicated 3D inverse kinematic problem with local data, which was first suggested and solved in linear approximation in [6]. In the full generality it was considered in [4], [5].

Definition of the *unsaturated* algorithms was introduced in [2]. About another approaches to inversion of the Radon transform and their generalizations, see [7] and [8], [9], [10] and references given there.

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